Coordinate Systems

Scott N. Walck

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1 Polar Coordinates

We will use the variables s and ϕ for polar coordinates. The coordinate s is the distance from the origin to a point in the plane, and the coordinate ϕ

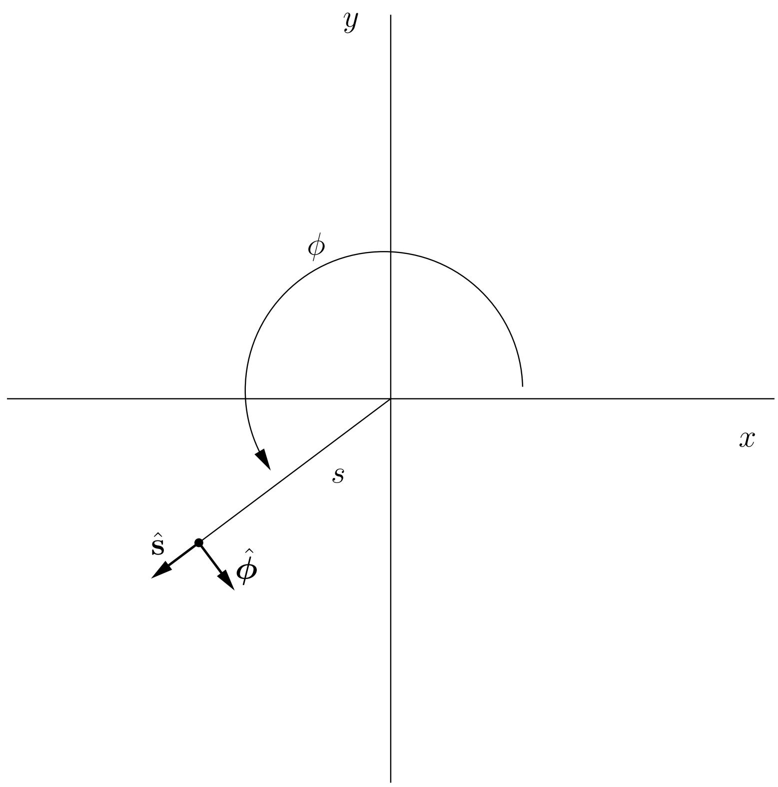

Figure 1: Polar coordinates.

is the angle between the x axis and a line joining the origin to a point. See Figure 1.

The Cartesian coordinates x and y are related to the polar coordinates s and ϕ by the following equations.

$$x = s\cos\phi$$
$$y = s\sin\phi$$

We introduce polar coordinate unit vectors. The unit vector $\hat{\mathbf{s}}$ points away from the origin. (This is a well-defined direction at every point in the plane except for the origin itself.) Equivalently, the unit vector $\hat{\mathbf{s}}$ points in the direction for which ϕ stays constant and s increases. Similarly, the unit vector $\hat{\boldsymbol{\phi}}$ points in the direction for which s stays constant and ϕ increases. We can write the polar coordinate unit vectors $\hat{\mathbf{s}}$ and $\hat{\boldsymbol{\phi}}$ in terms of the Cartesian coordinate unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as follows.

$$\hat{\mathbf{s}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2+y^2}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

Problem 1. Show that the polar coordinate unit vectors form an orthonormal system. In other words, show that

$$\hat{\mathbf{s}}\cdot\hat{\mathbf{s}} = 1$$
$$\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$$
$$\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\phi}} = 0.$$

Problem 2. Write $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in terms of $\hat{\mathbf{s}}$ and $\hat{\boldsymbol{\phi}}$.

Problem 3. Write $\frac{\partial}{\partial s}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. (You will need the vector calculus chain rule.)

Problem 4. Write $\frac{\partial}{\partial \phi}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Problem 5. Invert these expressions to give $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in terms of $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial \phi}$.

Unlike the Cartesian unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the polar unit vectors $\hat{\mathbf{s}}$ and $\hat{\boldsymbol{\phi}}$ point in different directions at different points in the plane.

Problem 6. Write $\frac{\partial\hat{\mathbf{s}}}{\partial s}$ and $\frac{\partial\hat{\mathbf{s}}}{\partial \phi}$ in terms of $\hat{\mathbf{s}}$ and $\hat{\boldsymbol{\phi}}$.

Problem 7. Write $\frac{\partial\hat{\boldsymbol{\phi}}}{\partial s}$ and $\frac{\partial\hat{\boldsymbol{\phi}}}{\partial \phi}$ in terms of $\hat{\mathbf{s}}$ and $\hat{\boldsymbol{\phi}}$.

1.1 2D Del Operator

There is an important vector operator called "del" and written ∇ that we will use over and over again in electromagnetic theory. In this section, we introduce a two-dimensional version of del for vector calculus in the plane. In Cartesian coordinates, del is written as follows.

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}$$

Problem 8. Write ∇ in polar coordinates, that is in terms of s, ϕ, $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, $\frac{\partial}{\partial s}$, and $\frac{\partial}{\partial \phi}$.

1.2 2D Gradient

The gradient of a scalar field is obtained by acting on the scalar field with the "del" operator ∇. Suppose f is a scalar field. In Cartesian coordinates,

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}.$$

Problem 9. Write the gradient of a scalar field $f(s, \phi)$ in polar coordinates.

1.3 2D Divergence

The divergence is the dot product of the vector operator ∇ with a vector field. If $\mathbf{A}$ is a vector field, the divergence of $\mathbf{A}$ is written $\nabla \cdot \mathbf{A}$.

If the vector field $\mathbf{A}$ is expressed in Cartesian coordinates,

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}},$$

then the divergence can be written

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}.$$

Consider a vector field in two dimensions, expressed in polar coordinates.

$$\mathbf{A}(s, \phi) = A_s(s, \phi)\hat{\mathbf{s}} + A_\phi(s, \phi)\hat{\boldsymbol{\phi}}$$

Problem 10. Write an expression for the divergence of $\mathbf{A}$ in polar coordinates.

1.4 2D Laplacian

The Laplacian is the divergence of the gradient. As such, we could write it $\nabla \cdot \nabla$, but we usually abbreviate it as ∇^2. In Cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Problem 11. Write the Laplacian of a scalar field $f(s, \phi)$ in polar coordinates.

2 Cylindrical Coordinates

We can use the cylindrical coordinates s, ϕ, and z to represent the location of a point in space. The coordinate s is the distance from the z axis to the point in space, the coordinate ϕ is the angle between the xz plane and the plane containing the z axis and the point, and the coordinate z means the same thing as in Cartesian coordinates (the distance from the xy plane). See Figure 2. Cylindrical coordinates are closely related to polar coordinates in that cylindrical coordinates describe the xy plane in a polar fashion, but continue to use the Cartesian z coordinate.

The Cartesian coordinates x, y, and z are related to the cylindrical coordinates s, ϕ, and z by the following equations.

$$\begin{aligned} x &= s \cos \phi \\ y &= s \sin \phi \\ z &= z \end{aligned}$$

We introduce cylindrical coordinate unit vectors. The unit vector $\hat{\mathbf{s}}$ points away from the z axis. (This is a well-defined direction at every point in space except for points on the z axis.) Equivalently, the unit vector $\hat{\mathbf{s}}$ points in the direction for which ϕ and z stay constant and s increases. Similarly, the unit vector $\hat{\boldsymbol{\phi}}$ points in the direction for which s and z stay constant and ϕ increases. Finally, the unit vector $\hat{\mathbf{z}}$ points in the direction for which s and ϕ stay constant and z increases. We can write the cylindrical coordinate unit vectors $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ in terms of the Cartesian coordinate unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$,

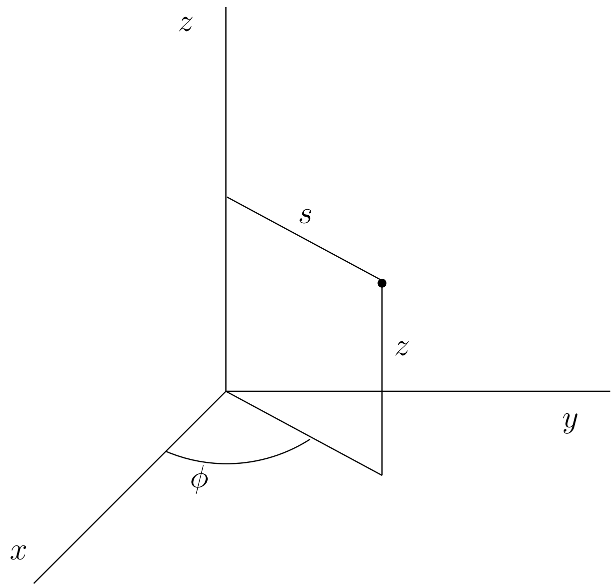

Figure 2: Cylindrical coordinates.

and $\hat{\mathbf{z}}$ as follows.

$$\hat{\mathbf{s}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Problem 12. Write $\frac{\partial\hat{\mathbf{s}}}{\partial s}$, $\frac{\partial\hat{\mathbf{s}}}{\partial\phi}$, and $\frac{\partial\hat{\mathbf{s}}}{\partial z}$ in terms of $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

Problem 13. Write $\frac{\partial\hat{\boldsymbol{\phi}}}{\partial s}$, $\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi}$, and $\frac{\partial\hat{\boldsymbol{\phi}}}{\partial z}$ in terms of $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

Problem 14. Write $\frac{\partial\hat{\mathbf{z}}}{\partial s}$, $\frac{\partial\hat{\mathbf{z}}}{\partial\phi}$, and $\frac{\partial\hat{\mathbf{z}}}{\partial z}$ in terms of $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

2.1 Del Operator

There is an important vector operator called "del" and written ∇ that we will use over and over again in electromagnetic theory. In Cartesian coordinates, del is written as follows.

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Problem 15. Write ∇ in cylindrical coordinates, that is in terms of s, ϕ, z, $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$, $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial \phi}$, and $\frac{\partial}{\partial z}$.

2.2 Gradient

The gradient of a scalar field is obtained by acting on the scalar field with the "del" operator ∇. Suppose f is a scalar field. In Cartesian coordinates,

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}.$$

Problem 16. Write the gradient of a scalar field $f(s, \phi, z)$ in cylindrical coordinates.

2.3 Divergence

The divergence is the dot product of the vector operator ∇ with a vector field. If $\mathbf{A}$ is a vector field, the divergence of $\mathbf{A}$ is written $\nabla \cdot \mathbf{A}$.

If the vector field $\mathbf{A}$ is expressed in Cartesian coordinates,

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}},$$

then the divergence can be written

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

Consider a vector field expressed in cylindrical coordinates.

$$\mathbf{A}(s, \phi, z) = A_s(s, \phi, z)\hat{\mathbf{s}} + A_\phi(s, \phi, z)\hat{\boldsymbol{\phi}} + A_z(s, \phi, z)\hat{\mathbf{z}}$$

Problem 17. Write an expression for the divergence of $\mathbf{A}$ in cylindrical coordinates.

2.4 Curl

The curl is the cross product of the vector operator ∇ with a vector field. If $\mathbf{A}$ is a vector field, the curl of $\mathbf{A}$ is written $\nabla \times \mathbf{A}$.

Consider a vector field expressed in cylindrical coordinates.

$$\mathbf{A}(s, \phi, z) = A_s(s, \phi, z)\hat{\mathbf{s}} + A_\phi(s, \phi, z)\hat{\boldsymbol{\phi}} + A_z(s, \phi, z)\hat{\mathbf{z}}$$

Problem 18. Write an expression for the curl of $\mathbf{A}$ in cylindrical coordinates.

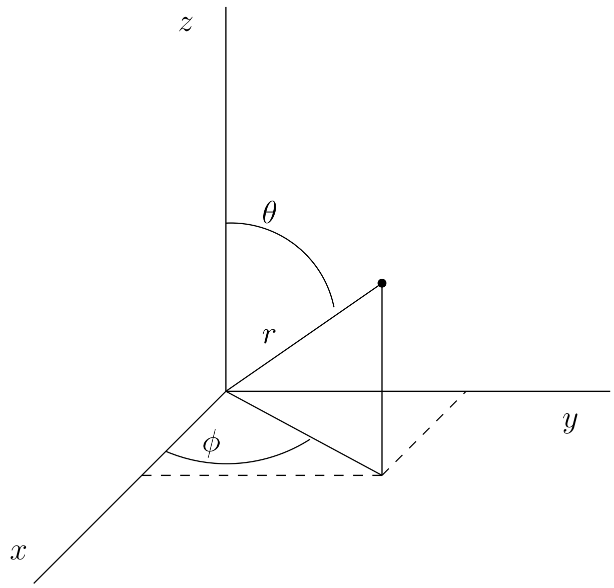

Figure 3: Spherical coordinates.

2.5 Laplacian

The Laplacian is the divergence of the gradient. As such, we could write it $\nabla \cdot \nabla$, but we usually abbreviate it as ∇^2.

Problem 19. Write the Laplacian of a scalar field $f(s, \phi, z)$ in cylindrical coordinates.

3 Spherical Coordinates

We can use the spherical coordinates r, θ, and ϕ to represent the location of a point in space. The coordinate r is the distance from the origin to the point in space, the coordinate θ is the angle between the z axis and a line from the origin to the point, and the coordinate ϕ is the angle between the xz plane and the plane containing the z axis and the point. (The coordinate ϕ has the same meaning in spherical coordinates that it does in cylindrical coordinates.) See Figure 3.

The Cartesian coordinates x, y, and z are related to the spherical coordinates r, θ, and ϕ by the following equations.

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

We introduce spherical coordinate unit vectors. The unit vector $\hat{\mathbf{r}}$ points away from the origin. (This is a well-defined direction at every point in space except for the origin itself.) Equivalently, the unit vector $\hat{\mathbf{r}}$ points in the direction for which θ and ϕ stay constant and r increases. Similarly, the unit vector $\hat{\boldsymbol{\theta}}$ points in the direction for which r and ϕ stay constant and θ increases. Finally, the unit vector $\hat{\boldsymbol{\phi}}$ points in the direction for which r and θ stay constant and ϕ increases. To write $\hat{\mathbf{r}}$ in terms of the Cartesian unit vectors, we divide the position vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ by its magnitude $r = \sqrt{x^2 + y^2 + z^2}$. The expression for $\hat{\boldsymbol{\phi}}$ is the same as it was for cylindrical coordinates. An expression for $\hat{\boldsymbol{\theta}}$ can be found from $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}}$. We can write the spherical coordinate unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ in terms of the Cartesian coordinate unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ as follows.

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

Problem 20. Show that the spherical coordinate unit vectors form an orthonormal system. In other words, show that

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$$
$$\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = 1$$
$$\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$$
$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$$
$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$$
$$\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = 0.$$

Problem 21. Write $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

Problem 22. Write $\frac{\partial}{\partial r}$ in terms of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$.

Problem 23. Write $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$.

Problem 24. Write $\frac{\partial}{\partial \phi}$ in terms of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$.

Problem 25. Invert these expressions to give $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ in terms of $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$, and $\frac{\partial}{\partial \phi}$.

Problem 26. Write $\frac{\partial \hat{\mathbf{r}}}{\partial r}$, $\frac{\partial \hat{\mathbf{r}}}{\partial \theta}$, and $\frac{\partial \hat{\mathbf{r}}}{\partial \phi}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

Problem 27. Write $\frac{\partial \hat{\boldsymbol{\theta}}}{\partial r}$, $\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta}$, and $\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

Problem 28. Write $\frac{\partial \hat{\boldsymbol{\phi}}}{\partial r}$, $\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta}$, and $\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

3.1 Del Operator

Problem 29. Write ∇ in spherical coordinates.

3.2 Gradient

Problem 30. Write the gradient of a scalar field $f(r, \theta, \phi)$ in spherical coordinates.

3.3 Divergence

Problem 31. Write an expression for the divergence of $\mathbf{A}$ in spherical coordinates.

3.4 Laplacian

Problem 32. Write the Laplacian of a scalar field $f(r, \theta, \phi)$ in spherical coordinates.