Purcell/Morin Notes

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These notes are for the third edition of Purcell and Morin.

Chapter 1

Electrostatics: charges and fields

1.1 Electric charge

1.2 Conservation of charge

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1.4 Coulomb's law

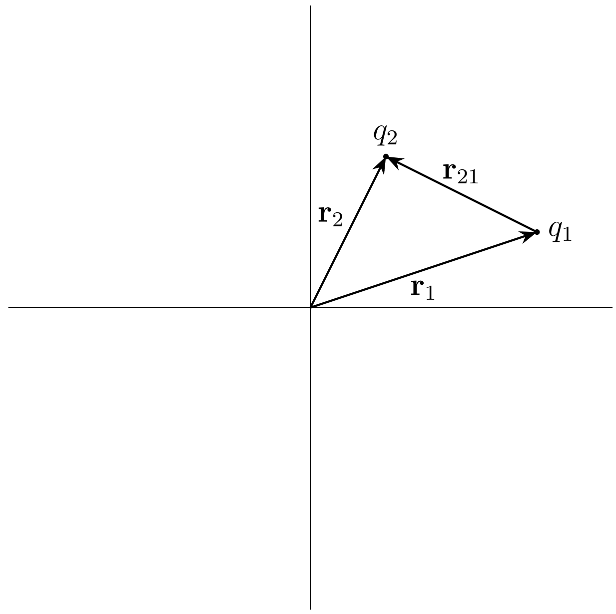
Equation (1.1), the first equation in the book, gives Coulomb's law.

$$\mathbf{F}_2 = k\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2}$$

The setting for Coulomb's law is that we have two point charges. The vector $\mathbf{r}_{21}$ is the displacement vector from charge 1 to charge 2, and $\mathbf{F}_2$ is the force on charge 2. The first thing we are going to do is replace k with $\frac{1}{4\pi\epsilon_0}$, since k is an overused symbol in physics. After the replacement, we arrive at equation (1.4) for Coulomb's law.

$$\mathbf{F}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2}$$

Note that equation (1.4) is the force on charge 2 (even though Purcell dropped the subscript). It is *not* the force on charge 1. The force on charge 1 has the same magnitude, but the opposite direction.



When we write r_{21} without bold font (or without an arrow above it when writing by hand), we mean the magnitude of the vector $\mathbf{r}_{21}$.

$$r_{21} = |\mathbf{r}_{21}|$$

When we write $\hat{\mathbf{r}}_{21}$, we mean the unit vector formed by dividing $\mathbf{r}_{21}$ by its magnitude.

$$\hat{\mathbf{r}}_{21} = \frac{\mathbf{r}_{21}}{r_{21}}$$

Once we choose a coordinate system, we can talk about the "position vectors" $\mathbf{r}_1$ and $\mathbf{r}_2$. Strictly speaking, position is not a vector because it makes no sense to add positions. When we say "position vector of particle 1", we mean displacement vector from the origin to particle 1. Note the relationship between $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_{21}$.

$$\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$$

This relationship comes from the picture above, since vectors add by putting them tail to tip.

$$\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}_{21}$$

Here is Coulomb's law written in terms of the position vectors.

$$\mathbf{F}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 \hat{\mathbf{r}}_{21}}{r_{21}^2} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{r}_{21}}{r_{21}^3} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

Example 1. Suppose $\mathbf{r}_1 = 3 \text{ m } \hat{\mathbf{x}} + 1 \text{ m } \hat{\mathbf{y}}$ and $\mathbf{r}_2 = 1 \text{ m } \hat{\mathbf{x}} + 2 \text{ m } \hat{\mathbf{y}}$. Then

$$\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1 = -2 \text{ m } \hat{\mathbf{x}} + 1 \text{ m } \hat{\mathbf{y}}$$

$$r_{21} = \sqrt{5} \text{ m}$$

$$\hat{\mathbf{r}}_{21} = -\frac{2}{\sqrt{5}}\hat{\mathbf{x}} + \frac{1}{\sqrt{5}}\hat{\mathbf{y}}$$

1.5 Energy of a system of charges

1.6 Electrical energy in a crystal lattice

1.7 The electric field

Sections 1.7 and 1.8 deal with calculating the electric field produced by a charge distribution.

1.7.1 Point charges

To calculate the electric field produced by one or more point charges, use equation (1.20):

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} \frac{q_j \hat{\mathbf{r}}_{0j}}{r_{0j}^2}$$

Remember that we have a bunch of notation for what is going on here. First, we have the source points $\mathbf{r}_j$. These are the places (N of them all together) where point charges are located.

$$\mathbf{r}_j = x_j\hat{\mathbf{x}} + y_j\hat{\mathbf{y}} + z_j\hat{\mathbf{z}}$$

Second, we have the field point $\mathbf{r}_0$. This is the place where we want to know the electric field.

$$\mathbf{r}_0 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

Note that the (x, y, z) that appear behind $\mathbf{E}$ in equation (1.20) are the Cartesian coordinates of the field point. Third, we have an extremely important displacement vector $\mathbf{r}_{0j}$ that points from source point $\mathbf{r}_j$ to the field point $\mathbf{r}_0$.

This vector doesn't really have a name, but it deserves one. We could call it the "source-to-field displacement vector". There will be a source-to-field displacement vector for each source charge.

$$\mathbf{r}_{0j} = \mathbf{r}_0 - \mathbf{r}_j$$

Fourth, we have the source-to-field distance r_{0j}, which is the magnitude of the source-to-field displacement vector.

$$r_{0j} = |\mathbf{r}_{0j}|$$

Fifth, we have the source-to-field unit vector $\hat{\mathbf{r}}_{0j}$, a unit vector that points from the source point toward the field point.

$$\hat{\mathbf{r}}_{0j} = \frac{\mathbf{r}_{0j}}{r_{0j}}$$

In typeset accounts such as these notes and the textbook, a boldface character means a vector. In handwritten accounts, we denote a vector by putting an arrow over the symbol. It is very important whether or not an arrow is over the symbol, because the notation means one thing with an arrow and another thing without an arrow.

1.8 Charge distributions

1.8.1 Volume charge

To calculate the electric field produced by a volume charge, use equation (1.22):

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x', y', z')\hat{\boldsymbol{\mathscr{r}}}\, dx'\, dy'\, dz'}{\mathscr{r}^2}$$

Purcell and Morin use $\mathbf{r}$ as the source-to-field displacement vector in equation (1.22). I do not like that choice of notation, because the vector $\mathbf{r}$ is already overused, and it makes more sense as a field point position. Therefore, I have used a script symbol $\boldsymbol{\mathscr{r}}$ to represent the source-to-field displacement vector, following the notation of David Griffiths.

The integral in (1.22) is to be taken over all of the source points. We use the vector $\mathbf{r}'$ to denote one of the source points.

$$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}$$

The field point $\mathbf{r}_0$ is the same as it was for point charges.

$$\mathbf{r}_0 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

The source-to-field displacement vector is now called $\boldsymbol{\mathscr{r}}$.

$$\boldsymbol{\mathscr{r}} = \mathbf{r}_0 - \mathbf{r}'$$

The choice by Purcell to use $\mathbf{r}$ as the source-to-field displacement vector is not the most super-awesome, because $\mathbf{r}$ ought to be equal to

$$x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}},$$

but the latter is not the source-to-field displacement vector.

The source-to-field distance $\mathscr{r}$ is the magnitude of the source-to-field displacement vector.

$$\mathscr{r} = |\boldsymbol{\mathscr{r}}| = |\mathbf{r}_0 - \mathbf{r}'|$$

The source-to-field unit vector $\hat{\boldsymbol{\mathscr{r}}}$ is a unit vector that points from the source point toward the field point.

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r}_0 - \mathbf{r}'}{|\mathbf{r}_0 - \mathbf{r}'|}$$

1.8.2 Line charge

What if we have charge distributed over a line or a curve, rather than throughout a volume? In this case, we need an equation like (1.22), but with a one-dimensional integral rather than a three-dimensional volume integral. Here is what we need.

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(x', y', z')\hat{\boldsymbol{\mathscr{r}}}\, dl'}{\mathscr{r}^2}$$

Here, λ is the linear charge density (charge per unit length), which can depend on location (so it is a function of x', y', and z'). The l' in dl' is not a real integration variable yet. What dl' turns into depends on the situation.

Example 1

Suppose that charge is distributed along the y axis from $-L$ to L. Then $\mathbf{r}' = y'\hat{\mathbf{y}}$ and $dl' = dy'$.

Example 2

Suppose that charge is distributed over a circle of radius R in the xy plane, centered on the origin. Then

$$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} = R\cos\phi\hat{\mathbf{x}} + R\sin\phi\hat{\mathbf{y}}$$

and

$$dl' = Rd\phi.$$

1.8.3 Surface charge

If we have charge distributed over a surface, we need an equation like (1.22), but with a two-dimensional surface integral rather than a three-dimensional volume integral. Here is what we need.

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(x', y', z')\hat{\boldsymbol{\mathscr{r}}}\,da'}{\mathscr{r}^2}$$

Here, σ is the surface charge density (charge per unit area), which can depend on location (so it is a function of x', y', and z'). The da' is a surface element whose form depends on the situation.

1.9 Flux

Example 2. Find the flux of the electric field

$$\mathbf{E} = Ayz\hat{\mathbf{x}} + Axz\hat{\mathbf{y}} + Axy\hat{\mathbf{z}}$$

through the rectangle in the yz plane with vertices $(0, 0, 0)$, $(0, a, 0)$, $(0, a, 4a)$, and $(0, 0, 4a)$, where a is a constant with dimensions of length, and A is a constant with appropriate dimensions. The orientation of the rectangle is $\hat{\mathbf{x}}$.

Solution: We'll start with the general expression for electric flux, equation (1.26).

$$\Phi = \int \mathbf{E}\cdot d\mathbf{a}$$

We need an expression for the area element $d\mathbf{a}$. We will want to integrate over y and z. The orientation should be in the $\hat{\mathbf{x}}$ (rather than $-\hat{\mathbf{x}}$) direction.

$$d\mathbf{a} = dy\,dz\hat{\mathbf{x}}$$

$$\mathbf{E} \cdot d\mathbf{a} = Ayz\, dy\, dz$$

$$\Phi = \int_0^{4a} \int_0^a Ayz\, dy\, dz = A \frac{a^2}{2} \frac{16a^2}{2} = 4Aa^4$$

Example 3. Find the electric flux produced by a point charge q at the origin through a cylindrical surface of radius R and height $2L$ centered at the origin. The cylindrical surface consists only of the "round part" with area $2\pi R(2L)$; it does not include a disk at the top or the bottom of the cylinder. Take the orientation of the surface to be "outward".

Solution: The definition of electric flux is

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a},$$

so we need expressions for $\mathbf{E}$ and $d\mathbf{a}$. The electric field produced by a point charge at the origin can be found from equation (1.20) and is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2}.$$

To write the surface element $d\mathbf{a}$, I want to use cylindrical coordinates. I will use cylindrical coordinates (s, ϕ, z) in which s is the distance from the z axis and ϕ is the angle with the positive x axis.

$$d\mathbf{a} = (s\, d\phi)(dz)\hat{\mathbf{s}} = s\, d\phi\, dz \hat{\mathbf{s}}$$

At this point, I have the electric field in spherical coordinates and the surface element in cylindrical coordinates. I want to use cylindrical coordinates to do the integral, so I will rewrite the electric field in cylindrical coordinates.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3} = \frac{q}{4\pi\epsilon_0} \frac{z\hat{\mathbf{z}} + s\hat{\mathbf{s}}}{(z^2 + s^2)^{3/2}}$$

$$\mathbf{E} \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0} \frac{s}{(z^2 + s^2)^{3/2}} s\, d\phi\, dz$$

The entire integration takes place at $s = R$, so we can replace s with R.

$$\Phi = \frac{qR^2}{4\pi\epsilon_0} \int_{-L}^{L} \int_0^{2\pi} \frac{d\phi\, dz}{(z^2 + R^2)^{3/2}} = \frac{qR^2}{2\epsilon_0} \int_{-L}^{L} \frac{dz}{(z^2 + R^2)^{3/2}}$$

We can look this integral up in our handy table of integrals.

$$\Phi = \frac{qR^2}{2\epsilon_0} \frac{1}{R^2} \frac{z}{\sqrt{z^2 + R^2}} \bigg|_{-L}^{L} = \frac{q}{\epsilon_0} \frac{L}{\sqrt{L^2 + R^2}}$$

Example 4. Find the electric flux produced by a point charge q at the origin through a disk of radius R lying in the plane $z = L$ and centered on the z axis. Assume L is positive and take the orientation of the surface to be $\hat{\mathbf{z}}$.

Solution: The definition of electric flux is

$$\Phi = \int \mathbf{E} \cdot d\mathbf{a},$$

so we need expressions for $\mathbf{E}$ and $d\mathbf{a}$. The electric field produced by a point charge at the origin can be found from equation (1.20) and is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

To write the surface element $d\mathbf{a}$ for the disk, I want to use cylindrical coordinates.

$$d\mathbf{a} = (ds)(s\,d\phi)\hat{\mathbf{z}} = s\,ds\,d\phi\hat{\mathbf{z}}$$

We'll use the expression for $\mathbf{E}$ in cylindrical coordinates from the previous problem.

$$\mathbf{E} \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\frac{z}{(z^2+s^2)^{3/2}} s\,ds\,d\phi$$

The entire integration takes place at $z = L$, so we can replace z with L.

$$\Phi = \frac{qL}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{s\,ds\,d\phi}{(L^2+s^2)^{3/2}} = \frac{qL}{2\epsilon_0}\int_0^R \frac{s\,ds}{(L^2+s^2)^{3/2}}$$

This integral is in the form of $u^n\,du$, so we can do it.

$$\Phi = \frac{qL}{4\epsilon_0}\left.\frac{(L^2+s^2)^{-1/2}}{-1/2}\right|_0^R = \frac{qL}{2\epsilon_0}\left[L^{-1} - (L^2+R^2)^{-1/2}\right]$$

$$\Phi = \frac{q}{2\epsilon_0}\left(1 - \frac{L}{\sqrt{L^2+R^2}}\right)$$

1.10 Gauss's law

1.11 Field of a spherical charge distribution

1.12 Field of a line charge

1.13 Field of an infinite flat sheet of charge

Equation (1.40) gives the magnitude of the electric field produced by an infinite flat sheet of charge. Since electric field is a vector, it is nicer to give a vector expression for the electric field. The electric field produced at a point $\mathbf{r}_0 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ by an infinite flat sheet of charge with surface charge density σ lying in the xy plane is

$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}} & , \text{ if } z > 0 \\ -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}} & , \text{ if } z < 0 \end{cases} .$$

Note that the electric field does not depend on x or y, but it does depend on z.

Equation (1.41) applies to a much broader set of situations than equation (1.40) and the equation above. Equation (1.41) applies to *any* surface with surface charge density σ. Equation (1.41) comes from Gauss's law, and it talks about the discontinuity in the component of the electric field that is perpendicular to the surface. If we write this equation as

$$\mathbf{E}_{\perp,P} - \mathbf{E}_{\perp,P'} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

where P is very close to the surface on one side, and P' is very close to the surface on the other side of the surface, then $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the surface, from P' toward P.

1.14 The force on a layer of charge

1.15 Energy associated with the electric field

1.16 Applications