

Vector Integrals

Scott N. Walck

October 13, 2016

Contents

1	A Table of Vector Integrals	2
2	Applications of the Integrals	2
2.1	Scalar Line Integral	2
2.1.1	Finding Total Charge of a Line Charge	2
2.1.2	Finding Electric Potential of a Line Charge	3
2.2	Vector Line Integral	3
2.2.1	Finding Electric Field of a Line Charge	3
2.3	Dotted Line Integral	3
2.3.1	Finding Electric Potential from Electric Field	3
2.4	Scalar Surface Integral	4
2.4.1	Finding Total Charge of a Surface Charge	4
2.4.2	Finding Electric Potential of a Surface Charge	4
2.5	Vector Surface Integral	4
2.5.1	Finding Electric Field of a Surface Charge	4
2.6	Flux Integral	4
2.6.1	Finding Electric Flux from Electric Field	5
2.7	Scalar Volume Integral	5
2.7.1	Finding Total Charge of a Volume Charge	5
2.7.2	Finding Electric Potential of a Volume Charge	5
2.8	Vector Volume Integral	5
2.8.1	Finding Electric Field of a Volume Charge	5

3	Fundamental Theorems of Calculus	6
3.1	Gradient Theorem	6
3.2	Stokes' Theorem	6
3.3	Divergence Theorem	6
4	Calculation	6
4.1	Line Integrals	6
4.1.1	Vector Line Integrals	7
4.1.2	Dotted Line Integrals	9
4.2	Flux Integrals	12

1 A Table of Vector Integrals

scalar line integral	$\int f dl$	1D
vector line integral	$\int \mathbf{F} dl$	1D
dotted line integral	$\int \mathbf{F} \cdot d\mathbf{l}$	1D
scalar surface integral	$\int f da$	2D
vector surface integral	$\int \mathbf{F} da$	2D
flux integral	$\int \mathbf{F} \cdot d\mathbf{a}$	2D
scalar volume integral	$\int f dv$	3D
vector volume integral	$\int \mathbf{F} dv$	3D

2 Applications of the Integrals

2.1 Scalar Line Integral

The scalar line integral requires a scalar field f and a path P .

$$\int_P f dl$$

2.1.1 Finding Total Charge of a Line Charge

The path P is along the line charge.

$$Q = \int_P \lambda(\mathbf{r}') dl'$$

2.1.2 Finding Electric Potential of a Line Charge

The path P is along the line charge.

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_P \frac{\lambda(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dl'$$

2.2 Vector Line Integral

The vector line integral requires a vector field \mathbf{F} and a path P .

$$\int_P \mathbf{F} dl$$

2.2.1 Finding Electric Field of a Line Charge

The path P is along the line charge.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_P \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \lambda(\mathbf{r}') dl'$$

2.3 Dotted Line Integral

The dotted line integral requires a vector field \mathbf{F} and a path P .

$$\int_P \mathbf{F} \cdot d\mathbf{l}$$

2.3.1 Finding Electric Potential from Electric Field

The path P begins at a point where the electric potential is zero and ends at the field point \mathbf{r} .

$$\phi(\mathbf{r}) = - \int_P \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'$$

In electrostatics, this integral is path independent, so we could write the integral using only the endpoints of the path P . Let \mathbf{a}_0 be a point we choose for the electric potential to be zero.

$$\phi(\mathbf{r}) = - \int_{\mathbf{a}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'$$

A nice property of this latter form is that we see how the field point \mathbf{r} on the left is related to the field point \mathbf{r} on the right. We are finding the electric potential at the end point \mathbf{r} of the path P .

2.4 Scalar Surface Integral

The scalar surface integral requires a scalar field f and a surface S . The surface does not need an orientation.

$$\int_S f da$$

2.4.1 Finding Total Charge of a Surface Charge

The surface S is over the surface charge.

$$Q = \int_S \sigma(\mathbf{r}') da'$$

2.4.2 Finding Electric Potential of a Surface Charge

The surface S is over the surface charge.

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} da'$$

2.5 Vector Surface Integral

The vector surface integral requires a vector field \mathbf{F} and a surface S . The surface does not need an orientation.

$$\int_S \mathbf{F} da$$

2.5.1 Finding Electric Field of a Surface Charge

The surface S is over the surface charge.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \sigma(\mathbf{r}') da'$$

2.6 Flux Integral

The flux integral requires a vector field \mathbf{F} and an oriented surface S .

$$\int_S \mathbf{F} \cdot d\mathbf{a}$$

2.6.1 Finding Electric Flux from Electric Field

The surface S is the surface through which to find the electric flux. The orientation points perpendicular to the surface, so that electric field in that direction would count as positive electric flux.

$$\Phi_E = \int_S \mathbf{E}(\mathbf{r}') \cdot d\mathbf{a}'$$

2.7 Scalar Volume Integral

The scalar volume integral requires a scalar field f and a volume V .

$$\int_V f dv$$

2.7.1 Finding Total Charge of a Volume Charge

The volume V is over the volume charge.

$$Q = \int_V \rho(\mathbf{r}') dv'$$

2.7.2 Finding Electric Potential of a Volume Charge

The volume V is over the volume charge.

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv'$$

2.8 Vector Volume Integral

The vector volume integral requires a vector field \mathbf{F} and a volume V .

$$\int_V \mathbf{F} dv$$

2.8.1 Finding Electric Field of a Volume Charge

The volume V is over the volume charge.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}') dv'$$

3 Fundamental Theorems of Calculus

3.1 Gradient Theorem

The gradient theorem requires a scalar field f and a path P . Suppose the path P starts at \mathbf{a} and ends at \mathbf{b} .

$$\int_P \nabla f \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

The integral on the left is a dotted line integral.

3.2 Stokes' Theorem

Stokes' theorem requires a vector field \mathbf{F} and an oriented surface S . Let ∂S be a (closed) path that forms the boundary of the surface S .

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{l}$$

The integral on the left is a flux integral, and the integral on the right is a dotted line integral.

3.3 Divergence Theorem

The divergence theorem requires a vector field \mathbf{F} and a volume V . Let ∂V be the (closed) oriented surface that forms the boundary of the volume V . The orientation is "outward".

$$\int_V (\nabla \cdot \mathbf{F}) dv = \int_{\partial V} \mathbf{F} \cdot d\mathbf{a}$$

The integral on the left is a scalar volume integral, and the integral on the right is a flux integral.

4 Calculation

4.1 Line Integrals

The key to evaluating line integrals, whether they be scalar line integrals, vector line integrals, or dotted line integrals, is to find a single variable or

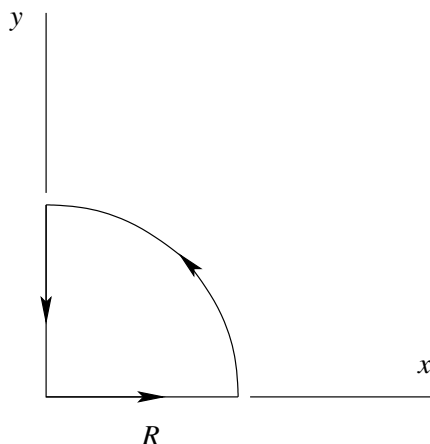
parameter in which to carry out the integration. In many cases, this single parameter can be one of the coordinates in a Cartesian, cylindrical, or spherical coordinate system.

4.1.1 Vector Line Integrals

Example 1. Find the vector line integral $\int \mathbf{F} dl$ for the vector field

$$\mathbf{F}(s, \phi, z) = s^2 \cos \phi \hat{\mathbf{s}} + s^2 \sin \phi \hat{\boldsymbol{\phi}}$$

over the closed path shown below.



Solution: The path is made up of three portions. Let P_1 be the portion that runs along the x axis. Let P_2 be the portion that curves, and let P_3 be the portion that runs along the y axis. Denote by P the entire closed path. The vector line integral over the entire path is the sum of the vector line integrals over each portion.

$$\int_P \mathbf{F} dl = \int_{P_1} \mathbf{F} dl + \int_{P_2} \mathbf{F} dl + \int_{P_3} \mathbf{F} dl$$

We will carry the integrals out in cylindrical coordinates. However, we want to avoid having the unit vectors $\hat{\mathbf{s}}$ and $\hat{\boldsymbol{\phi}}$ under the integral sign, because these unit vectors change direction from place to place. Therefore, we will rewrite the vector field \mathbf{F} using Cartesian unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, but keeping the cylindrical coordinates s , ϕ , and z .

We can write the cylindrical coordinate unit vectors $\hat{\mathbf{s}}$, $\hat{\phi}$, and $\hat{\mathbf{z}}$ in terms of the Cartesian coordinate unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ as follows.

$$\begin{aligned}\hat{\mathbf{s}} &= \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ \hat{\phi} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}\end{aligned}$$

The vector field \mathbf{F} is the following.

$$\begin{aligned}\mathbf{F} &= s^2 \cos \phi \hat{\mathbf{s}} + s^2 \sin \phi \hat{\phi} \\ &= s^2 \cos \phi (\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}) + s^2 \sin \phi (-\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}) \\ &= s^2 \cos 2\phi \hat{\mathbf{x}} + s^2 \sin 2\phi \hat{\mathbf{y}}\end{aligned}$$

Here, we have used the following trigonometric identities.

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta\end{aligned}$$

For the first portion P_1 , we have $dl = ds$.

$$\int_{P_1} \mathbf{F} dl = \int_0^R (s^2 \cos 2\phi \hat{\mathbf{x}} + s^2 \sin 2\phi \hat{\mathbf{y}}) ds$$

We want to get all variables in terms of s , since that is the one variable we are integrating over. Along P_1 , we have $\phi = 0$.

$$\int_{P_1} \mathbf{F} dl = \hat{\mathbf{x}} \int_0^R s^2 ds = \frac{R^3}{3} \hat{\mathbf{x}}$$

For the second portion P_2 , we have $dl = s d\phi$.

$$\int_{P_2} \mathbf{F} dl = \int_0^{\pi/2} (s^2 \cos 2\phi \hat{\mathbf{x}} + s^2 \sin 2\phi \hat{\mathbf{y}}) s d\phi$$

Along P_2 , we have $s = R$.

$$\begin{aligned}\int_{P_2} \mathbf{F} dl &= R^3 \int_0^{\pi/2} (\cos 2\phi \hat{\mathbf{x}} + \sin 2\phi \hat{\mathbf{y}}) d\phi \\ &= R^3 \hat{\mathbf{x}} \int_0^{\pi/2} \cos 2\phi d\phi + R^3 \hat{\mathbf{y}} \int_0^{\pi/2} \sin 2\phi d\phi\end{aligned}$$

$$\int_0^{\pi/2} \cos 2\phi \, d\phi = \left[\frac{\sin 2\phi}{2} \right]_0^{\pi/2} = 0$$

$$\int_0^{\pi/2} \sin 2\phi \, d\phi = \left[-\frac{\cos 2\phi}{2} \right]_0^{\pi/2} = \left[\frac{\cos 2\phi}{2} \right]_{\pi/2}^0 = \frac{1 - (-1)}{2} = 1$$

$$\int_{P_2} \mathbf{F} \, dl = R^3 \hat{\mathbf{y}}$$

For the third portion P_3 , we have $dl = ds$.

$$\int_{P_3} \mathbf{F} \, dl = \int_R^0 (s^2 \cos 2\phi \hat{\mathbf{x}} + s^2 \sin 2\phi \hat{\mathbf{y}}) \, ds$$

This looks very much like the expression for the integral over the first portion above, but it's different in two ways. First, the limits on the integral are different. For P_3 , we're starting at $s = R$ and going to $s = 0$. Second, the value of ϕ is different. For P_3 , we have $\phi = \pi/2$.

$$\int_{P_3} \mathbf{F} \, dl = -\hat{\mathbf{x}} \int_R^0 s^2 \, ds = \frac{R^3}{3} \hat{\mathbf{x}}$$

In total we have the following result.

$$\int_P \mathbf{F} \, dl = \frac{2}{3} R^3 \hat{\mathbf{x}} + R^3 \hat{\mathbf{y}}$$

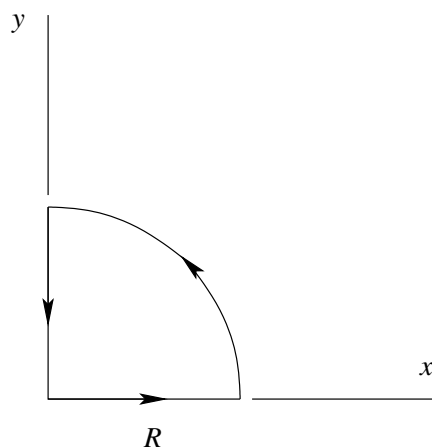
Note that the result of a vector line integral is a vector.

4.1.2 Dotted Line Integrals

Example 2. Find the dotted line integral $\int \mathbf{F} \cdot d\mathbf{l}$ for the vector field

$$\mathbf{F}(s, \phi, z) = s^2 \cos \phi \hat{\mathbf{s}} + s^2 \sin \phi \hat{\boldsymbol{\phi}}$$

over the closed path shown below.



Solution: The path is made up of three portions. Let P_1 be the portion that runs along the x axis. Let P_2 be the portion that curves, and let P_3 be the portion that runs along the y axis. Denote by P the entire closed path. The dotted line integral over the entire path is the sum of the dotted line integrals over each portion.

$$\int_P \mathbf{F} \cdot d\mathbf{l} = \int_{P_1} \mathbf{F} \cdot d\mathbf{l} + \int_{P_2} \mathbf{F} \cdot d\mathbf{l} + \int_{P_3} \mathbf{F} \cdot d\mathbf{l}$$

We will carry the integrals out in cylindrical coordinates. For $d\mathbf{l}$, we can use the standard expression in cylindrical coordinates.

$$d\mathbf{l} = ds\hat{\mathbf{s}} + s d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}}$$

Now we can find an expression for $\mathbf{F} \cdot d\mathbf{l}$.

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{l} &= (s^2 \cos \phi \hat{\mathbf{s}} + s^2 \sin \phi \hat{\boldsymbol{\phi}}) \cdot (ds\hat{\mathbf{s}} + s d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}}) \\ &= s^2 \cos \phi ds + s^3 \sin \phi d\phi \end{aligned}$$

Recall that for line integrals of all kinds, we want to express the integral in terms of a single variable that we are integrating over. The expression above is not yet in that form. We have three portions of the path that we want to integrate over, and it will turn out that the variable we want for our single variable of integration is different for the different portions. The expression for $\mathbf{F} \cdot d\mathbf{l}$ will simplify in different ways for each portion of the path.

For the first portion P_1 , we have $\phi = 0$. This means that $d\phi = 0$ and that part of $\mathbf{F} \cdot d\mathbf{l}$ will go away. For P_1 , we have

$$\mathbf{F} \cdot d\mathbf{l} = s^2 ds.$$

$$\int_{P_1} \mathbf{F} \cdot d\mathbf{l} = \int_0^R s^2 ds = \frac{R^3}{3}$$

For the second portion P_2 , we have $s = R$, $ds = 0$, and

$$\mathbf{F} \cdot d\mathbf{l} = R^3 \sin \phi d\phi.$$

$$\int_{P_2} \mathbf{F} \cdot d\mathbf{l} = \int_0^{\pi/2} R^3 \sin \phi d\phi = R^3$$

For the third portion P_3 , we have $\phi = \pi/2$, $d\phi = 0$, and

$$\mathbf{F} \cdot d\mathbf{l} = 0.$$

$$\int_{P_3} \mathbf{F} \cdot d\mathbf{l} = 0$$

In total we have the following result.

$$\int_P \mathbf{F} \cdot d\mathbf{l} = \frac{4}{3}R^3$$

Note that the result of a dotted line integral is a scalar.

4.2 Flux Integrals

Example 3. Find the flux integral for the vector field

$$\mathbf{F}(r, \theta, \phi) = r\hat{\mathbf{r}} + r \sin \theta \hat{\boldsymbol{\theta}} + r \sin \theta \cos \phi \hat{\boldsymbol{\phi}}$$

through the region enclosed by a triangle in the xy plane with vertices at $(x, y, z) = (0, 0, 0)$, $(x, y, z) = (2, 0, 0)$, and $(x, y, z) = (2, 2, 0)$.

Solution: The flux integral is

$$\int \mathbf{F} \cdot d\mathbf{a}$$

so the first order of business is to get an expression for $d\mathbf{a}$ in terms of the two variables we want to integrate over. The triangular region given suggests

that we want to use Cartesian coordinates and integrate over x and y . The surface element $d\mathbf{a}$ is a vector, so we need an orientation for the surface. The orientation is always perpendicular to the surface, so it could be either $\hat{\mathbf{z}}$ or $-\hat{\mathbf{z}}$. An orientation was not given in the problem, so let's choose $\hat{\mathbf{z}}$ for our orientation.

$$d\mathbf{a} = dx dy \hat{\mathbf{z}}$$

Next, we want an expression for $\mathbf{F} \cdot d\mathbf{a}$ in terms of the two variables we want to integrate over. We notice that the vector field is given in spherical coordinates and the surface is described with Cartesian coordinates. When we take the dot product, we will have terms such as $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}$ showing up. Let's figure these out in advance. Starting with expressions for the spherical unit vectors in terms of the Cartesian unit vectors,

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}\end{aligned}$$

we take the dot product of each of these equations with $\hat{\mathbf{z}}$ to find

$$\begin{aligned}\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} &= \cos \theta \\ \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{z}} &= -\sin \theta \\ \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} &= 0.\end{aligned}$$

Now we write an expression for $\mathbf{F} \cdot d\mathbf{a}$.

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{a} &= (r\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} + r \sin \theta \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{z}} + r \sin \theta \cos \phi \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}}) dx dy \\ &= (r \cos \theta - r \sin^2 \theta) dx dy \\ &= \left(z - \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + z^2}} \right) dx dy \\ &= -\sqrt{x^2 + y^2} dx dy\end{aligned}$$

In the last step, we used the fact that $z = 0$ for the surface we are using.

Next, we need limits for our integral.

$$\int \mathbf{F} \cdot d\mathbf{a} = - \int_0^2 \int_0^x \sqrt{x^2 + y^2} dy dx$$

The inner integral goes from the line $y = 0$ to the line $y = x$. The outer integral goes from the point $x = 0$ to the point $x = 2$.

All that remains is to evaluate the double integral. The inner integral is

$$\int_0^x \sqrt{x^2 + y^2} dy$$

This is an integral over y with x held constant. From Gradshteyn and Ryzhik, 5th edition, integral 2.271.3 (page 105) I find this integral.

$$\int \sqrt{a + x^2} dx = \frac{1}{2}x\sqrt{a + x^2} + \frac{1}{2}a \ln(x + \sqrt{a + x^2})$$

For a , we'll substitute x^2 , and for x we'll substitute y .

$$\begin{aligned} \int_0^x \sqrt{x^2 + y^2} dy &= \left[\frac{1}{2}y\sqrt{x^2 + y^2} + \frac{1}{2}x^2 \ln(y + \sqrt{x^2 + y^2}) \right]_{y=0}^{y=x} \\ &= \left[\frac{1}{2}x\sqrt{x^2 + x^2} + \frac{1}{2}x^2 \ln(x + \sqrt{x^2 + x^2}) \right] - \left[\frac{1}{2}x^2 \ln(\sqrt{x^2}) \right] \\ &= \frac{\sqrt{2}}{2}x^2 + \frac{1}{2}x^2 \ln(x + \sqrt{2}x) - \frac{1}{2}x^2 \ln x \\ &= \frac{\sqrt{2}}{2}x^2 + \frac{1}{2}x^2 \ln(1 + \sqrt{2}) \\ &= \frac{\sqrt{2} + \ln(1 + \sqrt{2})}{2}x^2 \end{aligned}$$

We used the fact that x is everywhere nonnegative on our surface so that

$$\sqrt{x^2} = x.$$

$$\begin{aligned}\int \mathbf{F} \cdot d\mathbf{a} &= - \int_0^2 \int_0^x \sqrt{x^2 + y^2} dy dx \\ &= - \int_0^2 \frac{\sqrt{2} + \ln(1 + \sqrt{2})}{2} x^2 dx \\ &= - \frac{8\sqrt{2} + \ln(1 + \sqrt{2})}{3} \\ &= - \frac{4}{3} [\sqrt{2} + \ln(1 + \sqrt{2})] \approx -3.06\end{aligned}$$

The negative result makes sense because the vector field points downward (in the $-z$ direction) in the xy plane.