Quantum Mechanics I (PHY 421)
Fall 2012

Exam 3

Problem 1 Consider a system composed of zero or one photons in an interferometer (basis $\{|u\rangle, |l\rangle, |0\rangle\}$) and a two-level atom (basis $\{|E_0\rangle, |E_1\rangle\}$).

(a) [4 points] A state of the composite system is given by

$$\begin{aligned}|\psi\rangle =& \frac{2}{5}|u, E_0\rangle + \frac{1}{5}|l, E_0\rangle + \frac{2}{5}|u, E_1\rangle \\ &+ \frac{8}{15}|l, E_1\rangle + \frac{4}{15}|0, E_1\rangle + \frac{8}{15}|0, E_0\rangle\end{aligned}$$

Is this state entangled, or is it a product state? Explain how you know.

(b) [4 points] Suppose that the system is described by state $|\psi\rangle$ from part (a). If a measurement of the photon is made, and the photon is found to be in the lower beam, what is the conditional state of the two-level atom?

(c) [4 points] The interferometer by itself (in the absence of the two-level atom) evolves according to the Hamiltonian $\mathsf{H}_{\text{field}}$ where

$$\mathsf{H}_{\text{field}} \left| u \right\rangle = \hbar\omega_0 \left| u \right\rangle , \quad \mathsf{H}_{\text{field}} \left| l \right\rangle = \hbar\omega_0 \left| l \right\rangle , \quad \mathsf{H}_{\text{field}} \left| 0 \right\rangle = 0,$$

and ω_0 is the angular frequency of the photon. We call the Hamiltonian $\mathsf{H}_{\text{field}}$ the *field Hamiltonian* because it describes the state of the quantum electromagnetic field (whether there is zero or one photon, and which beam contains a photon). Write the Hamiltonian $\mathsf{H}_{\text{field}}$ in outer product form.

(d) [4 points] The two-level atom by itself (in the absence of the interferometer) evolves according to the Hamiltonian H_{atom} where

$$\mathsf{H}_{\text{atom}} \left| E_k \right\rangle = E_k \left| E_k \right\rangle .$$

Write the Hamiltonian H_{atom} in outer product form.

(e) [4 points] Suppose now we place the two-level atom in the lower beam of the interferometer. Now there is an interaction between the electromagnetic field (zero or one photon) and the two-level atom. In particular, if there is a photon in the lower beam, it can be absorbed by the atom, exciting the atom to its higher energy level. We can model this interaction with an interaction Hamiltonian

$$\mathsf{H}_{\text{int}} = \frac{\hbar\Omega}{2} \left(|0, E_1\rangle \langle l, E_0| + |l, E_0\rangle \langle 0, E_1| \right).$$

The strength of the interaction is given by the parameter Ω. Roughly speaking, this interaction Hamiltonian says that if there is a photon in the lower beam and the atom is in its ground state, the photon might get absorbed by the atom, resulting in no photon in the lower beam, and an atom in its excited state. The reverse process may also occur: if the atom is in an excited state and the lower beam has no photon, the atom may emit a photon into the lower beam and drop to its ground state.

The full Hamiltonian for the system composed of zero or one photons in an interferometer (basis $\{|u\rangle, |l\rangle, |0\rangle\}$) and a two-level atom (basis $\{|E_0\rangle, |E_1\rangle\}$). is

$$\mathsf{H} = \mathsf{H}_{\text{field}} \otimes I + I \otimes \mathsf{H}_{\text{atom}} + \mathsf{H}_{\text{int}}.$$

Of the six basis states in the product basis, some of these are eigenstates of H and some are not. Which basis states are eigenstates of H and which are not? For those basis states that are eigenstates of H, give the associated energies.

(f) [4 points] Consider the basis states that were *not* eigenstates of H. Some linear combinations of these basis states must be eigenstates of H. Consider the special case in which $\hbar\omega_0 = E_1 - E_0$ (in this case the photon energy matches the difference in energy levels of the atom, and the linear combinations are particularly nice). Find the remaining eigenstates of H and their associated eigenvalues by either (a) guess and check, or (b) making a matrix and finding the eigenvalues and eigenvectors (you should not need to work with anything larger than a 2×2 matrix).