Quantum Mechanics I (PHY 421)
Fall 2013

Exam 3

Problem 1 Consider a system composed of zero or one photons in an interferometer (basis $\{|u\rangle, |l\rangle, |0\rangle\}$) and a two-level atom (basis $\{|E_0\rangle, |E_1\rangle\}$). The interferometer by itself (in the absence of the two-level atom) evolves according to the Hamiltonian $\mathsf{H}_{\text{field}}$ where

$$\mathsf{H}_{\text{field}} |u\rangle = \hbar\omega_0 |u\rangle, \qquad \mathsf{H}_{\text{field}} |l\rangle = \hbar\omega_0 |l\rangle, \qquad \mathsf{H}_{\text{field}} |0\rangle = 0,$$

and ω_0 is the angular frequency of the photon. The two-level atom by itself (in the absence of the interferometer) evolves according to the Hamiltonian H_{atom} where

$$\mathsf{H}_{\text{atom}} |E_k\rangle = E_k |E_k\rangle .$$

Let us also assume that $\hbar\omega_0 = E_1 - E_0$, so that the energy of a photon is just right to make a transition in the two-level atom. Suppose now we place the two-level atom in the lower beam of the interferometer. Now there is an interaction between the electromagnetic field (zero or one photon) and the two-level atom. In particular, if there is a photon in the lower beam, it can be absorbed by the atom, exciting the atom to its higher energy level. We can model this interaction with an interaction Hamiltonian

$$\mathsf{H}_{\text{int}} = \frac{\hbar\Omega}{2} \left(|0, E_1\rangle \langle l, E_0| + |l, E_0\rangle \langle 0, E_1| \right) .$$

The strength of the interaction is given by the parameter Ω. Roughly speaking, this interaction Hamiltonian says that if there is a photon in the lower beam and the atom is in its ground state, the photon might get absorbed by the atom, resulting in no photon in the lower beam, and an atom in its excited state. The reverse process may also occur: if the atom is in an excited state and the lower beam has no photon, the atom may emit a photon into the lower beam and drop to its ground state.

The full Hamiltonian for the system composed of zero or one photons in an interferometer (basis $\{|u\rangle, |l\rangle, |0\rangle\}$) and a two-level atom (basis $\{|E_0\rangle, |E_1\rangle\}$). is

$$\mathsf{H} = \mathsf{H}_{\text{field}} \otimes I + I \otimes \mathsf{H}_{\text{atom}} + \mathsf{H}_{\text{int}}.$$

(a) Find the energies and eigenstates of the full Hamiltonian. (Hint: Of the six basis states in the product basis, some of these are eigenstates of H and some are not. First find all of the product basis states that are eigenstates of H. Then find linear combinations of the remaining product basis states to complete your collection of eigenstates of H.)

(b) Find the time evolution operator $\mathsf{U}(t)$.

(c) Supposing that the system starts in the state

$$|\psi(0)\rangle = |l, E_0\rangle \,,$$

find $|\psi(t)\rangle$.

Problem 2 A quantum wave function is given by

$$\psi(x) = A \exp\left(-\frac{x^2}{2a^2}\right),$$

where A is a constant chosen to normalize $\psi(x)$. Find A. Then calculate $\langle x \rangle$, Δx, $\langle p \rangle$, and Δp, for this wave function. Verify the Heisenberg uncertainty principle.