Quantum Mechanics I (PHY 421)

Kets Homework

Problem 1 Suppose that

$$|\alpha\rangle = (1-2i)\,|z_+\rangle + (-3+4i)\,|z_-\rangle$$

and

$$|\beta\rangle = i\,|z_+\rangle - |z_-\rangle$$

Express the following as linear combinations of $|z_+\rangle$ and $|z_-\rangle$.

1. $|\alpha\rangle + |\beta\rangle$
2. $|\alpha\rangle - |\beta\rangle$
3. $(1+2i)\,|\alpha\rangle - 2i\,|\beta\rangle$

Problem 2 Given expressions for $|x_+\rangle$ and $|x_-\rangle$ in terms of $|z_+\rangle$ and $|z_-\rangle$,

$$|x_+\rangle = \frac{1}{\sqrt{2}}\,|z_+\rangle + \frac{1}{\sqrt{2}}\,|z_-\rangle\,,$$
$$|x_-\rangle = \frac{1}{\sqrt{2}}\,|z_+\rangle - \frac{1}{\sqrt{2}}\,|z_-\rangle\,,$$

solve for $|z_+\rangle$ and $|z_-\rangle$ in terms of $|x_+\rangle$ and $|x_-\rangle$.

Problem 3 Given expressions for $|y_+\rangle$ and $|y_-\rangle$ in terms of $|z_+\rangle$ and $|z_-\rangle$,

$$|y_+\rangle = \frac{1}{\sqrt{2}}\,|z_+\rangle + \frac{i}{\sqrt{2}}\,|z_-\rangle\,,$$
$$|y_-\rangle = \frac{1}{\sqrt{2}}\,|z_+\rangle - \frac{i}{\sqrt{2}}\,|z_-\rangle\,,$$

along with our previous expressions for $|x_+\rangle$ and $|x_-\rangle$ in terms of $|z_+\rangle$ and $|z_-\rangle$, solve for $|x_+\rangle$ and $|x_-\rangle$ in terms of $|y_+\rangle$ and $|y_-\rangle$.

Problem 4 Express the following bra vectors in terms of the bras $\langle z_+|$ and $\langle z_-|$. Use the definitions of $|\alpha\rangle$ and $|\beta\rangle$ from the problem above.

1. $\langle\alpha|$
2. $\langle\beta|$
3. the bra vector corresponding to the ket vector $(1+2i)|\alpha\rangle - 2i|\beta\rangle$
4. $(1+2i)\langle\alpha| - 2i\langle\beta|$

Problem 5 Find the following inner products. Use the definitions of $|\alpha\rangle$ and $|\beta\rangle$ from the problem above.

1. $\langle\alpha|\beta\rangle$
2. $\langle\beta|\alpha\rangle$
3. $\langle x_+|y_+\rangle$

Problem 6 Consider the following expression.

$$|n_+\rangle = \cos\frac{\theta}{2}|z_+\rangle + e^{i\phi}\sin\frac{\theta}{2}|z_-\rangle$$

1. For what values of θ and ϕ does $|n_+\rangle = |x_+\rangle$?
2. For what values of θ and ϕ does $|n_+\rangle = |x_-\rangle$?
3. For what values of θ and ϕ does $|n_+\rangle = |y_+\rangle$?
4. For what values of θ and ϕ does $|n_+\rangle = |y_-\rangle$?
5. For what values of θ and ϕ does $|n_+\rangle = |z_+\rangle$?
6. For what values of θ and ϕ does $|n_+\rangle = |z_-\rangle$?

Do these angles make sense in terms of three-dimensional geometry?

Problem 7 Consider the following expression.

$$|n_-\rangle = \sin\frac{\theta}{2}|z_+\rangle - e^{i\phi}\cos\frac{\theta}{2}|z_-\rangle$$

1. For what values of θ and ϕ does $|n_+\rangle = |x_+\rangle$ and $|n_-\rangle = |x_-\rangle$?
2. For what values of θ and ϕ does $|n_+\rangle = |y_+\rangle$ and $|n_-\rangle = |y_-\rangle$?

Problem 8 Show that

$$\begin{aligned}\langle n_+ | n_+ \rangle &= 1\\ \langle n_+ | n_- \rangle &= 0\\ \langle n_- | n_+ \rangle &= 0\\ \langle n_- | n_- \rangle &= 1.\end{aligned}$$

You have just shown that $\{|n_+\rangle, |n_-\rangle\}$ is an orthonormal basis for our vector space of spin-1/2 states. Notice that this also means that $\{|x_+\rangle, |x_-\rangle\}$ and $\{|y_+\rangle, |y_-\rangle\}$ are orthonormal bases, since these are special cases of $\{|n_+\rangle, |n_-\rangle\}$.

Problem 9 Normalize the following state kets.

1. $|z_+\rangle + |z_-\rangle$
2. $|z_+\rangle + (2+3i)|z_-\rangle$
3. $|z_+\rangle + |x_-\rangle$
4. $|x_+\rangle + |y_+\rangle + |z_+\rangle$
5. $\cos\frac{\theta}{2}|z_+\rangle$

Problem 10 Consider a spin-1/2 particle in the state

$$|\psi\rangle = \frac{1}{2}|z_+\rangle + \left(\frac{1}{2} - \frac{i}{\sqrt{2}}\right)|z_-\rangle$$

1. If a measurement of spin in the z-direction is made, what are the probabilities of obtaining spin up and spin down?
2. If instead a measurement of spin in the y-direction is made, what are the probabilities of obtaining spin up and spin down?

Problem 11 Consider a spin-1/2 particle in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|z_+\rangle + \frac{1}{2}|z_-\rangle$$

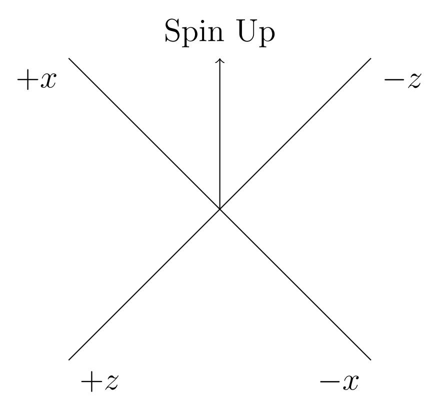
(a) If a measurement of spin in the x direction is made, what are the probabilities of obtaining spin up and spin down?

(b) If instead a measurement of spin in the y direction is made, what are the probabilities of obtaining spin up and spin down?

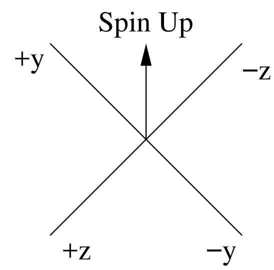
(c) If instead a measurement of spin in the z direction is made, what are the probabilities of obtaining spin up and spin down?

Problem 12 Write down a quantum state that has a probability of 1/3 of being found with spin up in the x-direction and a probability of 2/3 of being found with spin down in the x-direction.

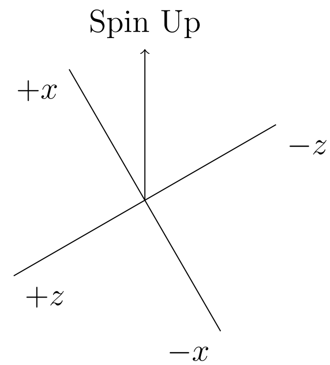
Problem 13 Bob constructs his Stern-Gerlach apparatus so that "spin up" corresponds to the direction in the xz-plane that is halfway between the positive x-axis and the negative z-axis. For which state is he guaranteed to measure spin up? (Express this state in terms of the standard $|z_+\rangle, |z_-\rangle$ basis.)



Problem 14 Bob constructs his Stern-Gerlach apparatus so that "spin up" corresponds to the direction in the yz-plane that is halfway between the positive y-axis and the negative z-axis. For which state is he guaranteed to measure spin up? (Express this state in terms of the standard $|z_+\rangle, |z_-\rangle$ basis.)



Problem 15 Bob constructs his Stern-Gerlach apparatus so that "spin up" corresponds to the direction in the xz-plane that makes and angle of 30° with the positive x axis and an angle of 60° with the negative z axis. For which state is he guaranteed to measure spin up? Express this state in terms of the standard $|z_+\rangle, |z_-\rangle$ basis.



Problem 16 Consider a two-level atom with ground state $|E_0\rangle$ and excited state $|E_1\rangle$. Consider the state $|u\rangle$ of the two-level atom given by

$$|u\rangle = \frac{4}{5}|E_0\rangle + \frac{3}{5}|E_1\rangle .$$

Suppose that $|u\rangle$ is the state associated with the outcome u of some measurement, and also that the initial state of the atom is $|\psi(0)\rangle = |u\rangle$. At time $t = 0$ the probability that our measurement would yield u is

$$P_u(0) = |\langle u|\, \psi(0)\rangle|^2 = 1.$$

Over time, however, the atom's state will evolve into something different. (a) Give an expression for the atom's state $|\psi(t)\rangle$ at a later time t. (b) Give an expression for the probability $P_u(t)$ of obtaining outcome u in a measurement made at time t.

Problem 17 If we put a spin-1/2 particle in a magnetic field, it can act like a two-level atom. In particular, the up and down spin states in the direction of the magnetic field are the energy level states. Consider a magnetic field in the z direction. The spin state $|z_+\rangle$ has energy $-\Omega\hbar/2$ and the spin state $|z_-\rangle$ has energy $\Omega\hbar/2$, where Ω is the Larmor frequency that depends on the strength of the magnetic field. Suppose now that we have a spin-1/2 particle in the initial state

$$|\psi(0)\rangle = \frac{4}{5}|x_+\rangle + \frac{3}{5}|x_-\rangle$$

It will not stay in the state $|\psi(0)\rangle$. Give an expression for the atom's state $|\psi(t)\rangle$ at a later time t.

Problem 18 If we put a spin-1/2 particle in a magnetic field, it can act like a two-level atom. In particular, the up and down spin states in the direction of the magnetic field are the energy level states. For a magnetic field in the x direction, the spin state $|x_+\rangle$ has energy $-\Omega\hbar/2$ and the spin state $|x_-\rangle$ has energy $\Omega\hbar/2$, where Ω is the Larmor frequency that depends on the strength of the magnetic field. Suppose now that we have a spin-1/2 particle in the initial state

$$|\psi(0)\rangle = \frac{3}{5}|z_+\rangle + \frac{4}{5}|z_-\rangle$$

It will not stay in the state $|\psi(0)\rangle$.

(a) Give an expression for the atom's state $|\psi(t)\rangle$ at a later time t.

(b) Over time, the probability for measuring spin up in the z direction may change. Find the largest and smallest probabilities of measuring spin in the z direction to be up.