Quick Reference Guide to Linear Algebra in Quantum Mechanics

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1 Complex Numbers

1.1 Introduction

Complex numbers are pairs of real numbers. If x and y are real numbers, then the pair (x, y) can be regarded as a complex number. Usually, we write this complex number as

$$x + yi.$$

We call the real number x the *real part* of the complex number and the real number y the *imaginary part* of the complex number.

Example: The numbers $\frac{1}{2}$ and $-\frac{\sqrt{3}}{2}$ are real numbers, so the pair $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ can be regarded as a complex number, which we will typically write as

$$\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

The complex number $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ can be written in other ways.

$$\frac{1}{2} - \frac{\sqrt{3}}{2}i = \frac{1}{2} - i\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}i + \frac{1}{2} = -i\frac{\sqrt{3}}{2} + \frac{1}{2}$$

1.2 Real Numbers

We regard the real numbers as a subset of the complex numbers. Real numbers are those complex numbers that have zero for their imaginary part. Instead of writing $4 + 0i$ for such a number, we will just write 4. In this way, every real number is a complex number.

Complex numbers can be added, subtracted, multiplied, and divided.

1.3 Addition of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, then the sum $z_1 + z_2$ is defined to be

$$z_1 + z_2 := (x_1 + x_2) + (y_1 + y_2)i.$$

The symbol ":=" means "is defined to be". To add two complex numbers, we just add the real parts and add the imaginary parts.

> **Example:** Let $z_1 = -3 - 4i$ and $z_2 = 5i - 7$. Then the sum is $z_1 + z_2 = -10 + i$.

1.4 Subtraction of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, then the difference $z_1 - z_2$ is defined to be

$$z_1 - z_2 := (x_1 - x_2) + (y_1 - y_2)i.$$

To subtract two complex numbers, we just subtract the real parts and subtract the imaginary parts.

> **Example:** Let $z_1 = -3 - 4i$ and $z_2 = 5i - 7$. Then the difference is $z_1 - z_2 = 4 - 9i$.

1.5 Multiplication of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, then the product $z_1 z_2$ is defined to be

$$z_1 z_2 := (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The best way to think about this is to imagine that you are distributing the product term by term, and using the idea that $i^2 = -1$.

$$\begin{aligned}(x_1 + y_1 i)(x_2 + y_2 i) &= x_1 x_2 + x_1 y_2 i + y_1 i x_2 + y_1 i y_2 i \\ &= x_1 x_2 + x_1 y_2 i + y_1 x_2 i - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i\end{aligned}$$

1.6 The Complex Conjugate

The *complex conjugate* of a complex number $z = x + yi$ is defined to be the complex number

$$z^* := x - yi.$$

Mathematicians usually write $\overline{z}$ for the complex conjugate of z, while physicists usually write z^*. We will use the physicists' convention in these notes. You can think of the complex conjugate as being formed by replacing every i in a complex number by $-i$.

If you multiply any complex number by its complex conjugate, you get a real number.

$$(x + yi)(x - yi) = x^2 + y^2$$

1.7 Division of Complex Numbers

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, and $z_2 \neq 0$, then we can form a quotient. We can simplify the quotient by multiplying the numerator and denominator by the complex conjugate of the denominator.

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{x_1 + y_1 i}{x_2 + y_2 i} \\ &= \frac{x_1 + y_1 i}{x_2 + y_2 i}\left(\frac{x_2 - y_2 i}{x_2 - y_2 i}\right) \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{-x_1 y_2 + y_1 x_2}{x_2^2 + y_2^2} i\end{aligned}$$

1.8 The Complex Plane

Since complex numbers are pairs of real numbers, it makes sense to think of them as points on a plane. By convention, we use the horizontal axis to represent the real part of a complex number, and the vertical axis to represent the imaginary part.

1.9 Magnitude of a Complex Number

The *magnitude* of a complex number is the distance from the origin on the complex plane. Therefore, the magnitude of a complex number must always be a nonnegative real number. The magnitude will be zero if and only if the complex number itself is zero. If $z = x + yi$ is a complex number, we write $|z|$ to denote the magnitude of z, and

$$|z| = \sqrt{x^2 + y^2}$$

from the Pythagorean theorem.

1.10 Exponential Function

If $w = u + vi$ is a complex number (with u and v real numbers), we define

$$e^w = e^{u+vi} = e^u(\cos v + i \sin v)$$

where v is understood as an angle in radians. This is really just a fancy version of the Euler formula, which holds if θ is real.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

1.11 Polar Form of a Complex Number

Consider the complex number $z = Re^{i\theta}$, where R and θ are real numbers, and R is nonnegative. Using the Euler formula, we can write this as

$$z = Re^{i\theta} = R(\cos\theta + i\sin\theta) = R\cos\theta + iR\sin\theta$$

This complex number has real part $R\cos\theta$ and imaginary part $R\sin\theta$. This is just the point in the complex plane with polar coordinates R and θ! Notice that R is the magnitude of the complex number. The angle θ is called the *argument* of the complex number z.

Every complex number z can be written in *Cartesian form*

$$z = x + yi$$

with x and y real, as well as *polar form*

$$z = Re^{i\theta}$$

with R and θ real. The relationship between these two forms for writing complex numbers is the same as the relationship between Cartesian and polar coordinates.

$$x = R\cos\theta \qquad R = \sqrt{x^2 + y^2}$$
$$y = R\sin\theta \qquad \theta = \arctan\frac{y}{x}$$

1.12 Multiplication in Polar Form

Multiplication of complex numbers is especially easy to do if they are written in polar form.

$$R_1e^{i\theta_1}R_2e^{i\theta_2} = R_1R_2e^{i(\theta_1+\theta_2)}$$

To form the product of complex numbers in polar form you multiply the magnitudes and add the arguments.

1.13 Problems

Try to avoid decimal computation in the following exercises. (For example, write $\sqrt{2}$ instead of 1.41.....)

Problem 1 *Simplify the following and express in Cartesian form.*

1. $(4e^{i\pi/2})(4e^{i\pi})$
2. $4e^{i\pi/2} + 4e^{i\pi}$
3. $3e^{i\pi/4} + 4e^{i3\pi/4}$
4. $(1 + i)(\sqrt{3} - i)$

Problem 2 *Simplify the following and express in polar form.*

1. $(4e^{i\pi/2})(4e^{i\pi})$
2. $4e^{i\pi/2} + 4e^{i\pi}$
3. $(1+i)(\sqrt{3}-i)$
4. $(e^{i\pi/6})(e^{i\pi/9}) + (2e^{i3\pi/2})(e^{-i2\pi/9})$

Problem 3 *Find* i^*i, $(-i)^*(-i)$, $|i|^2$, $|-i|^2$, $|i|$, $|-i|$.

Problem 4 *Let* $z = 3+4i$. *Find* z^*, z^*z, $|z|$, z^2, *and* $1/z$.

Problem 5 *Is it always true that* $|z|^2 = z^*z$ *? Show that it's true or find a counterexample. (Hint: If you want to try to show that it's true, write* $z = x+iy$ *with* x *and* y *real.)*

Problem 6 *Find two different values for* z *so that* $z^2 = i$. *Express these numbers in both Cartesian and polar form. Plot them on the complex plane.*

Problem 7 *Show, on the complex plane, all of the complex numbers with magnitude 1.*

Problem 8 *Simplify* $|e^{i\omega t} + e^{-i\omega t}|^2$, *where* ω *and* t *are real numbers. Is the result a real number? If so, write it in a way that does not have an* i *in it.*

2 Kets

A *ket* (or ket vector) is a symbol consisting of a vertical bar, a descriptive label, and a right angle bracket, as in $|\phi_1\rangle$.

We can add two (or more) kets to form another ket. For example, we write

$$|\psi\rangle = |\phi_1\rangle + |\phi_2\rangle\,.$$

Ket addition is *commutative*, meaning that the order in which addition is done does not matter.

$$|\phi_1\rangle + |\phi_2\rangle = |\phi_2\rangle + |\phi_1\rangle$$

(Later, we will see that some kinds of multiplication that we do are not commutative. All kinds of addition that we do are commutative.)

We can multiply a ket by a complex number to form another ket. For example, we might write

$$|\psi\rangle = (3 + 4i)\,|\phi_1\rangle\,.$$

This multiplication of a ket by a complex number is also commutative. For example,

$$(3 + 4i)\,|\phi_1\rangle = |\phi_1\rangle\,(3 + 4i).$$

It is conventional, but not required, to write complex numbers in front of the kets they multiply, rather than behind.

Multiplication of two kets is *not* defined.

Let c_1 and c_2 be complex numbers and $|\phi_1\rangle$ and $|\phi_2\rangle$ be kets. We can distribute (complex) numbers over a ket,

$$(c_1 + c_2)\,|\phi_1\rangle = c_1\,|\phi_1\rangle + c_2\,|\phi_1\rangle$$

and we can distribute a (complex) number over kets,

$$c_1(|\phi_1\rangle + |\phi_2\rangle) = c_1\,|\phi_1\rangle + c_1\,|\phi_2\rangle\,.$$

A set V of kets is called a *ket space* if it satisfies the following conditions.

1. There is a ket $|\text{zero}\rangle \in V$, such that $|\phi\rangle + |\text{zero}\rangle = |\phi\rangle$ for all $|\phi\rangle \in V$.
2. If $|\phi_1\rangle\,, |\phi_2\rangle \in V$, then $|\phi_1\rangle + |\phi_2\rangle \in V$.
3. If c is any complex number and $|\phi_1\rangle \in V$, then $c\,|\phi_1\rangle \in V$.

The ket $|\text{zero}\rangle$ is called the *zero ket*. It is the additive identity element of the ket space. You might think that a better notation for the zero ket would be $|0\rangle$, but it is conventional to reserve the symbol $|0\rangle$ for something else. We will not have a need to write down the symbol for the zero ket very often, so $|\text{zero}\rangle$ will be a fine notation for us. At some point, we may become slightly sloppy with our notation and use the symbol 0 (without ket notation) to represent the zero ket. Notice that not every set of kets is a ket space. The set must contain all sums and scalar multiples in order to be a ket space.

A *linear combination* of kets $|\phi_1\rangle\,, \ldots, |\phi_n\rangle$ is a (weighted) sum of these kets

$$c_1\,|\phi_1\rangle + \cdots + c_n\,|\phi_n\rangle$$

with complex coefficients $c_1, \ldots, c_n$. Any of the complex coefficients $c_1, \ldots, c_n$ could be zero.

The *span* of a set of kets $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ is the collection of all linear combinations of these kets. A set of kets is said to *span* a ket space V if the span of the set of kets contains V. Notice that we have two uses for the word *span*, one as a noun and one as a verb.

A set of kets is called *linearly dependent* if one of the kets in the set can be written as a linear combination of the other kets in the set. A set of kets that is not linearly dependent is called *linearly independent*.

A *basis* for the ket space V is a linearly independent set of kets that spans V. If $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ is a basis for the ket space V, then every ket in V can be written as a linear combination of $|\phi_1\rangle, \ldots, |\phi_n\rangle$ in a unique way.

The *dimension* of a ket space V is the smallest number of kets that span V. The dimension of V is equal to the number of kets in any basis for V. The dimension of a ket space may be finite or infinite. We work with finite-dimensional ket spaces in this Quick Reference Guide, but quantum mechanics needs infinite-dimensional ket spaces as well.

3 Bras

A *bra* (or bra vector) is a symbol consisting of a left angle bracket, a descriptive label, and a vertical bar, as in $\langle\phi_1|$. For each ket, there is a corresponding bra with the same descriptive label inside, and for each bra, there is a corresponding ket.

We can add two (or more) bras to form another bra. For example, we might write

$$\langle\psi| = \langle\phi_1| + \langle\phi_2|.$$

We can multiply a bra by a complex number to form another bra. For example, we might write

$$\langle\chi| = (3 + 4i)\langle\phi_1|.$$

Multiplication of two bras is not defined.

Let c_1 and c_2 be complex numbers and $\langle\phi_1|$ and $\langle\phi_2|$ be bras. We can distribute (complex) numbers over a bra,

$$(c_1 + c_2)\langle\phi_1| = c_1\langle\phi_1| + c_2\langle\phi_1|$$

and we can distribute a (complex) number over bras,

$$c_1(\langle\phi_1| + \langle\phi_2|) = c_1 \langle\phi_1| + c_1 \langle\phi_2| \,.$$

We said before that for each ket, there is a corresponding bra with the same descriptive label inside. If, in addition, there are coefficients multiplying the ket, we must take their complex conjugates in forming the corresponding bra. Here are some examples.

1. The ket $|\phi_1\rangle$ has corresponding bra $\langle\phi_1|$.
2. The ket $(3 + 4i)\,|\phi_1\rangle$ has corresponding bra $(3 - 4i)\,\langle\phi_1|$.
3. The ket $(3+4i)\,|\phi_1\rangle - 2i\,|\phi_2\rangle$ has corresponding bra $(3-4i)\,\langle\phi_1| + 2i\,\langle\phi_2|$.
4. The ket $c_1\,|\phi_1\rangle$ has corresponding bra $c_1^*\,\langle\phi_1|$.
5. The bra $c_2\,\langle\phi_2|$ has corresponding ket $c_2^*\,|\phi_2\rangle$.

In the expressions above, c_1^* is the complex conjugate of the complex number c_1.

If V is a ket space, we will call the collection of corresponding bras a *bra space*, and denote it V^*.

4 Inner Product

A bra multiplied by a ket (with the bra on the left and the ket on the right) is called an *inner product* and gives a complex number. For example, the inner product of $\langle\phi_1|$ and $|\psi_2\rangle$ is written $\langle\phi_1|\psi_2\rangle$, and is a complex number.

The inner product has the following properties. Let $|\phi_1\rangle$, $|\phi_2\rangle$ be kets, $\langle\chi_1|$, $\langle\chi_2|$ be bras, and c_1, c_2 be complex numbers.

1. $\langle\chi_1|\,(c_1\,|\phi_1\rangle + c_2\,|\phi_2\rangle) = c_1\,\langle\chi_1|\phi_1\rangle + c_2\,\langle\chi_1|\phi_2\rangle$
2. $(c_1\,\langle\chi_1| + c_2\,\langle\chi_2|)\,|\phi_1\rangle = c_1\,\langle\chi_1|\phi_1\rangle + c_2\,\langle\chi_2|\phi_1\rangle$
3. $\langle\chi_1|\phi_1\rangle = \langle\phi_1|\chi_1\rangle^*$

Note that the last property implies that $\langle\phi_1|\phi_1\rangle$ is real. In addition,

1. $\langle\phi_1|\phi_1\rangle \geq 0$

2. $\langle\phi_1|\phi_1\rangle = 0$ if and only if $|\phi_1\rangle = |\text{zero}\rangle$

Two kets are called *orthogonal* if their inner product is zero. What we really mean here is that the inner product between the first ket's corresponding bra and the second ket is zero. A set of kets is called orthogonal if any pair of distinct kets in the set is orthogonal.

The *norm*, or length, of a ket $|\psi\rangle$ is defined to be $\sqrt{\langle\psi|\psi\rangle}$.

A ket is called *normalized* if it has norm 1.

An *orthonormal set* of kets is an orthogonal set of normalized kets.

An *orthonormal basis* for a ket space V is an orthogonal set of normalized kets that forms a basis for V. If $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ is an orthonormal basis for the ket space V, then every ket in V can be written as a linear combination of $|\phi_1\rangle, \ldots, |\phi_n\rangle$ in a unique way, and

$$\langle\phi_l|\phi_m\rangle = \begin{cases} 1 & , \quad \text{if } l = m \\ 0 & , \quad \text{if } l \neq m \end{cases} .$$

This last condition is the orthonormality condition. It says that different kets in the basis are orthogonal to each other, and that individual kets in the basis are normalized. A short hand way to write this orthonormality condition is

$$\langle\phi_l|\phi_m\rangle = \delta_{lm}$$

where δ_{lm} is called the Kronecker delta, defined as

$$\delta_{lm} = \begin{cases} 1 & , \quad \text{if } l = m \\ 0 & , \quad \text{if } l \neq m \end{cases} .$$

The inner product gives us a new way to think about bra vectors. We can think of them as maps from a ket space to the complex numbers. In other words, a bra vector is a thing which is waiting to take a ket as input and give a complex number as output.

5 Linear Operators

In linear algebra, an *operator* is a thing that acts on a vector. For example, if we were working with two-dimensional real vectors in a plane, the command to "rotate by 120° counterclockwise" is an operator.

Let V be a ket space. A *linear operator* on V is a linear map $V \to V$. In other words, a linear operator is a function that takes a ket as input, gives a ket back as output, and does this in a linear way. We use a hat over a capital letter to denote a linear operator, such as $\hat{A}$. Because $\hat{A}$ is linear, we can distribute it over kets as follows.

$$\hat{A}\left(c_1 \left|\phi_1\right\rangle + c_2 \left|\phi_2\right\rangle\right) = c_1 \hat{A} \left|\phi_1\right\rangle + c_2 \hat{A} \left|\phi_2\right\rangle$$

So, if $\hat{A}$ is a linear operator and $|\phi\rangle$ is a ket, then $\hat{A}\,|\phi\rangle$ is a ket. An expression like $|\phi\rangle\,\hat{A}$ has no meaning. We must write linear operators to the left of the kets they act on. Since $\hat{A}\,|\phi\rangle$ is a ket, we can take the inner product of a bra $\langle\psi|$ with the ket $\hat{A}\,|\phi\rangle$ to obtain the complex number $\langle\psi|\,\hat{A}\,|\phi\rangle$.

In a similar way, we should regard an object like $\langle\psi|\,\hat{A}$ as a bra vector, since it is a thing which is waiting to turn a ket into a complex number. An expression like $\hat{A}\,\langle\psi|$ has no meaning.

Two linear operators on V are equal if they give the same result when acting on any ket in V. But, because any ket in V can be written as a linear combination of basis vectors of V, two linear operators on V are equal if they give the same result when acting on any basis ket.

We can define the sum of two operators on V. Let $\hat{A}$ and $\hat{B}$ be operators on V. The action of the sum, $\hat{A}+\hat{B}$, on any ket $|\psi\rangle \in V$ is defined to be the sum of the kets $\hat{A}\,|\psi\rangle$ and $\hat{B}\,|\psi\rangle$.

$$(\hat{A}+\hat{B})\,|\psi\rangle = \hat{A}\,|\psi\rangle + \hat{B}\,|\psi\rangle$$

We can define the product of two operators on V. Let $\hat{A}$ and $\hat{B}$ be operators on V. The action of the product, $\hat{A}\hat{B}$, on any ket $|\psi\rangle \in V$ is defined to be the action of $\hat{A}$ on the ket $\hat{B}\,|\psi\rangle$.

$$(\hat{A}\hat{B})\,|\psi\rangle = \hat{A}(\hat{B}\,|\psi\rangle)$$

While operator addition is commutative, so that

$$\hat{A}+\hat{B} = \hat{B}+\hat{A}$$

for all linear operators $\hat{A}$ and $\hat{B}$, operator multiplication is *not* commutative, and in general,

$$\hat{A}\hat{B} \neq \hat{B}\hat{A}.$$

5.1 The Commutator

We define the *commutator* of two linear operators, $\hat{A}$ and $\hat{B}$, to be

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

The commutator of two linear operators is another linear operator.

5.2 Outer Product

An *outer product* is a product of a ket on the left and a bra on the right. An outer product is a linear operator. For example, $|\phi_1\rangle \langle\phi_2|$ is a linear operator, because when it acts on a ket $|\psi\rangle$, it gives back a ket.

$$(|\phi_1\rangle \langle\phi_2|) |\psi\rangle = |\phi_1\rangle \langle\phi_2|\psi\rangle = \langle\phi_2|\psi\rangle |\phi_1\rangle$$

In the last equality, we brought the complex number $\langle\phi_2|\psi\rangle$ to the front of the expression.

5.3 Completeness Relation

Let $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ be an orthonormal basis for a ket space V. Then

$$\sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l| = \hat{I}, \tag{1}$$

where $\hat{I}$ is the identity operator on V. The identity operator is the linear operator that gives back the same ket that it acts on. In other words, for any $|\psi\rangle \in V$, we have $\hat{I}\,|\psi\rangle = |\psi\rangle$.

We can prove the completeness relation given above in the following way. Let $|\psi\rangle \in V$. Since $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ is an orthonormal basis for V, we can write $|\psi\rangle$ as a linear combination of basis kets.

$$|\psi\rangle = \sum_{m=1}^{n} c_m |\phi_m\rangle$$

Now,

$$\begin{aligned}
\left[\sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l|\right] |\psi\rangle &= \left[\sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l|\right] \sum_{m=1}^{n} c_m |\phi_m\rangle \\
&= \sum_{l=1}^{n} \sum_{m=1}^{n} c_m |\phi_l\rangle \langle\phi_l|\phi_m\rangle \\
&= \sum_{l=1}^{n} \sum_{m=1}^{n} c_m |\phi_l\rangle \delta_{lm} \\
&= \sum_{l=1}^{n} c_l |\phi_l\rangle \\
&= |\psi\rangle
\end{aligned}$$

In going from the third line to the fourth line, we used the fact that δ_{lm} gives zero unless $l = m$ to eliminate one of the sums and to replace c_m with c_l. This is an important little algebra trick that gets used in several places, and is good to understand.

We see that the operator $\sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l|$ does the same thing to kets that the identity operator does (namely, it leaves them alone), so it must be the identity operator.

5.4 Every Operator can be written as a sum of Outer Products

Let $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ be an orthonormal basis for a ket space V and let $\hat{A}$ be a linear operator on V. Using the completeness relation, $\hat{A}$ can be written as a sum of outer products of basis vectors.

$$
\begin{aligned}
\hat{A} &= \sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l| \hat{A} \sum_{m=1}^{n} |\phi_m\rangle \langle\phi_m| \\
&= \sum_{l=1}^{n} \sum_{m=1}^{n} \langle\phi_l| \hat{A} |\phi_m\rangle |\phi_l\rangle \langle\phi_m| \\
&= \sum_{l=1}^{n} \sum_{m=1}^{n} A_{lm} |\phi_l\rangle \langle\phi_m|
\end{aligned}
$$

where

$$
A_{lm} = \langle\phi_l| \hat{A} |\phi_m\rangle
$$

Notice that the set of numbers A_{lm} completely describes the linear operator $\hat{A}$ (given an orthonormal basis).

5.5 Eigenkets and Eigenvalues

If linear operator $\hat{A}$ acts on a non-zero ket $|\phi\rangle$ and returns a (complex) multiple of $|\phi\rangle$,

$$
\hat{A} |\phi\rangle = a |\phi\rangle
$$

(a a complex number), then $|\phi\rangle$ is called an *eigenket* of $\hat{A}$ and a is the corresponding *eigenvalue*.

6 Matrix Representation

6.1 Kets

Given a basis, we can represent a ket as a column vector. If $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$ is a basis for a ket space V, we associate a column vector of complex numbers with a ket as follows.

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \longleftrightarrow c_1 |\phi_1\rangle + \cdots + c_n |\phi_n\rangle$$

The complex numbers are just the coefficients of the kets, when expressed in terms of the given basis. Note that a basis is really an *ordered set*; the order of the kets in the basis matters.

If $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$ is an orthonormal basis for the ket space V, then every ket $|\psi\rangle \in V$ can be represented as a column vector of complex numbers as follows.

$$|\psi\rangle = \left[\sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l|\right] |\psi\rangle = \sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l|\psi\rangle \longleftrightarrow \begin{bmatrix} \langle\phi_1|\psi\rangle \\ \vdots \\ \langle\phi_n|\psi\rangle \end{bmatrix}$$

6.2 Bras

Given a basis, we can represent a bra as a row vector. If $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$ is a basis for a ket space V, we associate a row vector of complex numbers with a bra as follows.

$$\begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \longleftrightarrow c_1 \langle\phi_1| + \cdots + c_n \langle\phi_n|$$

If $\{|\phi_1\rangle, \dots, |\phi_n\rangle\}$ is an orthonormal basis for the ket space V, then every bra $\langle\psi| \in V^*$ can be represented as a row vector of complex numbers as follows.

$$\langle\psi| = \langle\psi| \left[\sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l|\right] = \sum_{l=1}^{n} \langle\psi|\phi_l\rangle \langle\phi_l| \longleftrightarrow \begin{bmatrix} \langle\psi|\phi_1\rangle & \dots & \langle\psi|\phi_n\rangle \end{bmatrix}$$

6.3 Linear Operators

Given a basis, we can represent a linear operator as a square matrix. If $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ is a basis for a ket space V, we associate a square matrix of complex numbers with a linear operator as follows.

$$\begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix} \longleftrightarrow \sum_{l=1}^{n}\sum_{m=1}^{n} A_{lm} |\phi_l\rangle \langle\phi_m|$$

If $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ is an orthonormal basis for the ket space V, then every linear operator $\hat{A}$ on V can be represented as a square matrix of complex numbers as follows.

$$\begin{aligned} \hat{A} &= \left[\sum_{l=1}^{n} |\phi_l\rangle \langle\phi_l|\right] \hat{A} \left[\sum_{m=1}^{n} |\phi_m\rangle \langle\phi_m|\right] \\ &= \sum_{l=1}^{n}\sum_{m=1}^{n} \langle\phi_l| \hat{A} |\phi_m\rangle |\phi_l\rangle \langle\phi_m| \longleftrightarrow \begin{bmatrix} \langle\phi_1| \hat{A} |\phi_1\rangle & \ldots & \langle\phi_1| \hat{A} |\phi_n\rangle \\ \vdots & \ddots & \vdots \\ \langle\phi_n| \hat{A} |\phi_1\rangle & \ldots & \langle\phi_n| \hat{A} |\phi_n\rangle \end{bmatrix} \end{aligned}$$

7 The Dagger

We have learned to regard an expression like $\hat{A}|\phi\rangle$ as a ket and an expression like $\langle\psi|\hat{B}$ as a bra. What is the bra that corresponds to the ket $\hat{A}|\phi\rangle$? The answer turns out to be $\langle\phi|\hat{A}^\dagger$, where $\hat{A}^\dagger$ is a linear operator that may not be the same as $\hat{A}$. For every linear operator $\hat{A}$, there is a linear operator $\hat{A}^\dagger$, called the *adjoint operator*.

The dagger operation turns kets into bras, bras into kets, complex numbers into their complex conjugates, and operators into their adjoints. In addition, it reverses the order of multiplication. Let's see some examples of the dagger operation.

1. $(|\phi_1\rangle)^\dagger = \langle\phi_1|$
2. $(\langle\phi_1|)^\dagger = |\phi_1\rangle$
3. $[(3+4i)|\phi_1\rangle]^\dagger = \langle\phi_1|(3-4i) = (3-4i)\langle\phi_1|$

4. $(\langle\psi|\phi\rangle)^\dagger = \langle\phi|\psi\rangle$
5. $(\langle\psi|\phi\rangle)^\dagger = \langle\psi|\phi\rangle^*$

In the fourth example, we regarded the expression on which the dagger acted as a product of a bra and a ket, and used the rule that the dagger turns kets into bras, bras into kets, and reverses order. In the fifth example, we regarded the expression as a complex number and used the rule that the dagger turns complex numbers into their complex conjugates. Fortunately, one of the properties of the inner product is that the expressions obtained in these two examples are equal. Here are some more examples.

1. $\left(\langle\psi|\,\hat{A}\,|\phi\rangle\right)^\dagger = \langle\phi|\,\hat{A}^\dagger\,|\psi\rangle$
2. $\left(\langle\psi|\,\hat{A}\,|\phi\rangle\right)^\dagger = \langle\psi|\,\hat{A}\,|\phi\rangle^*$
3. $(|\chi\rangle\,\langle\phi|)^\dagger = |\phi\rangle\,\langle\chi|$

7.1 Hermitian Operators

A linear operator $\hat{A}$ is called *Hermitian* if $\hat{A} = \hat{A}^\dagger$. The eigenvalues of a Hermitian operator are real.

7.2 Unitary Operators

A linear operator $\hat{A}$ is called *unitary* if $\hat{A}^\dagger\hat{A} = \hat{I}$. The eigenvalues of a unitary operator are complex numbers with magnitude one.

8 Spectral Decomposition

Theorem 1 (Spectral Theorem) *If $\hat{A}$ is a Hermitian linear operator acting on a ket space V, then there is an orthonormal basis of eigenvectors of $\hat{A}$. In other words, there is a collection of eigenvectors $|\phi_j\rangle$ (there will be n of them if we are working with an n-dimensional ket space) such that*

$$\hat{A}\,|\phi_j\rangle = a_j\,|\phi_j\rangle$$

and

$$\langle\phi_j|\phi_k\rangle = \delta_{jk}.$$

Suppose that $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$ is an orthonormal basis of eigenstates of $\hat{A}$. Then we have

$$\hat{A}\,|\phi_m\rangle = a_m\,|\phi_m\rangle$$

Using the completeness relation, we get

$$\hat{A} = \hat{A}\sum_{m=1}^{n} |\phi_m\rangle\langle\phi_m| = \sum_{m=1}^{n} a_m\,|\phi_m\rangle\langle\phi_m|\,.$$

8.1 Projection Operators

A *projection operator* is any linear operator $\hat{P}$ that satisfies

$$\hat{P}\hat{P} = \hat{P}.$$

Example: Let $|\chi\rangle$ be a normalized ket. Then $\hat{P} = |\chi\rangle\langle\chi|$ is a projection operator because

$$\hat{P}\hat{P} = |\chi\rangle\langle\chi|\chi\rangle\langle\chi| = |\chi\rangle\langle\chi| = \hat{P}.$$

Example: Let $|\chi_1\rangle$ and $|\chi_2\rangle$ be normalized kets that are orthogonal to each other. In other words, $\langle\chi_1|\chi_1\rangle = \langle\chi_2|\chi_2\rangle = 1$ and $\langle\chi_1|\chi_2\rangle = \langle\chi_2|\chi_1\rangle = 0$. Then $\hat{P} = |\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2|$ is a projection operator because

$$\begin{aligned}\hat{P}\hat{P} &= (|\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2|)(|\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2|)\\ &= |\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2| = \hat{P}.\end{aligned}$$

Now, let $\hat{A}$ be a Hermitian linear operator. By Theorem 1, there is an orthonormal basis $\{|\phi_j\rangle\}$ of eigenvectors of $\hat{A}$ with corresponding eigenvalues a_j.

$$\hat{A}\,|\phi_j\rangle = a_j\,|\phi_j\rangle$$

Corresponding to each eigenvalue a of $\hat{A}$ there is a projection operator $\hat{P}_a$ defined by

$$\hat{P}_a = \sum_{j|a_j=a} |\phi_j\rangle\langle\phi_j|$$

where the sum is over all eigenkets with eigenvalue a.

Recall that we have a completeness (or closure) relation

$$\sum_j |\phi_j\rangle \langle\phi_j| = \hat{I}.$$

With projection operators corresponding to each eigenvalue, we have a new way of writing that completion relation.

$$\sum_a \hat{P}_a = \hat{I}$$

Here, the sum is over all *distinct* eigenvalues of $\hat{A}$.

9 Tensor Product

Let V_A and V_B be ket spaces. We construct a new ket space $V_A \otimes V_B$ called the *tensor product* of the ket spaces V_A and V_B. If $\{|\phi_1\rangle, \ldots, |\phi_m\rangle\}$ is an orthonormal basis for V_A and $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ is an orthonormal basis for V_B, then the collection of the mn kets

$$\begin{gathered}
|\phi_1\rangle \otimes |\psi_1\rangle \\
\vdots \\
|\phi_1\rangle \otimes |\psi_n\rangle \\
|\phi_2\rangle \otimes |\psi_1\rangle \\
\vdots \\
|\phi_2\rangle \otimes |\psi_n\rangle \\
\vdots \\
|\phi_m\rangle \otimes |\psi_1\rangle \\
\vdots \\
|\phi_m\rangle \otimes |\psi_n\rangle
\end{gathered}$$

is an orthonormal basis for $V_A \otimes V_B$.

> **Example:** Suppose that V_A is a two-dimensional ket space with orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle\}$, and that V_B is a three-dimensional ket space with orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$. Then $V_A \otimes V_B$ is a six-dimensional ket space with orthonormal basis $\{|\phi_1\rangle \otimes |\psi_1\rangle, |\phi_1\rangle \otimes |\psi_2\rangle, |\phi_1\rangle \otimes |\psi_3\rangle, |\phi_2\rangle \otimes |\psi_1\rangle, |\phi_2\rangle \otimes |\psi_2\rangle, |\phi_2\rangle \otimes |\psi_3\rangle\}$.

The tensor product is distributive. For example,

$$\frac{1}{\sqrt{2}}\left|\phi_1\right\rangle \otimes \left|\psi_1\right\rangle + \frac{1}{\sqrt{2}}\left|\phi_1\right\rangle \otimes \left|\psi_2\right\rangle = \left|\phi_1\right\rangle \otimes \left(\frac{1}{\sqrt{2}}\left|\psi_1\right\rangle + \frac{1}{\sqrt{2}}\left|\psi_2\right\rangle\right).$$

The tensor product is *not* commutative.

$$\left|\phi_1\right\rangle \otimes \left|\psi_1\right\rangle \neq \left|\psi_1\right\rangle \otimes \left|\phi_1\right\rangle$$

10 Help Eliminate Errors

Accuracy and correctness are very important to me, as well as to readers of this document. If this document contains errors, I would like to fix them, and you can help me. Please let me know if you find spelling mistakes, mathematical mistakes, or any other mistakes. In honor of Donald Knuth, who set the standard for care in technical writing, I will happily pay $2.56 (one hexadecimal dollar) for each substantive or typographical error in this document to the first person who reports the error.

The following people have helped improve this document.

- David Walsh

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